Orthogonal Basis

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Outline

- Orthogonal/Orthonormal Basis
- Orthogonal Decomposition Theory
- How to find Orthonormal Basis

Reference: Textbook Chapter 7.2, 7.3

Orthogonal Set

- A set of vectors is called an orthogonal set if every pair of distinct vectors in the set is orthogonal.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$ An orthogonal set?

By definition, a set with only one vector is an orthogonal set.

Is orthogonal set independent?

Independent?

- Any orthogonal set of nonzero vectors is linearly independent.

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal set $\mathbf{v}_i \neq \mathbf{0}$ for $i = 1, 2, \ldots, k$.

Assume $c_1, c_2, \ldots, c_k$ make $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_i\mathbf{v}_i + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{v}_i$$

$$= \quad c_1\mathbf{v}_1 \cdot \mathbf{v}_i + c_2\mathbf{v}_2 \cdot \mathbf{v}_i + \cdots + c_i\mathbf{v}_i \cdot \mathbf{v}_i + \cdots + c_k\mathbf{v}_k \cdot \mathbf{v}_i$$

$$= \quad c_i(\mathbf{v}_i \cdot \mathbf{v}_i) \quad = \quad c_i \underbrace{||\mathbf{v}_i||^2}_{\neq 0} \Rightarrow c_i = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0$$

Orthonormal Set

- A set of vectors is called an orthonormal set if it is an orthogonal set, and the norm of all the vectors is 1

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$

$$\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \frac{1}{\sqrt{42}}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

Is orthonormal set independent?

Yes

A vector that has norm equal to 1 is called a unit vector.

Orthogonal Basis

- A basis that is an orthogonal (orthonormal) set is called an orthogonal (orthonormal) basis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonal basis of R^3

Orthonormal basis of R^3

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Orthogonal Basis

- Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal basis for a subspace W, and let u be a vector in W.

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$\frac{u \cdot v_1}{\|v_1\|^2} \quad \frac{u \cdot v_2}{\|v_2\|^2} \quad \frac{u \cdot v_k}{\|v_k\|^2}$$

Proof

How about orthonormal basis?

To find c_i

$$u \cdot v_i = (c_1 v_1 + c_2 v_2 + \cdots + c_i v_i + \cdots + c_k v_k) \cdot v_i$$

$$= c_1 v_1 \cdot v_i + c_2 v_2 \cdot v_i + \cdots + c_i v_i \cdot v_i + \cdots + c_k v_k \cdot v_i$$

$$= c_i (v_i \cdot v_i) = c_i \|v_i\|^2 \Rightarrow c_i = \frac{u \cdot v_i}{\|v_i\|^2}$$

Example

- Example: $\mathcal{S}$ = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is an orthogonal basis for $\mathcal{R}^3$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$\text{Let } \mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

c_1 c_2 c_3

Orthogonal Projection

- Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal basis for a subspace W, and let u be a vector in W.

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$\frac{u \cdot v_1}{\|v_1\|^2} \qquad \frac{u \cdot v_2}{\|v_2\|^2} \qquad\qquad \frac{u \cdot v_k}{\|v_k\|^2}$$

- Let u be any vector, and w is the orthogonal projection of u on W.

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$\frac{u \cdot v_1}{\|v_1\|^2} \qquad \frac{u \cdot v_2}{\|v_2\|^2} \qquad\qquad \frac{u \cdot v_k}{\|v_k\|^2}$$

Orthogonal Projection

- Let $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal basis for a subspace W. Let u be any vector, and w is the orthogonal projection of u on W.

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$\frac{u \cdot v_1}{\|v_1\|^2} \quad \frac{u \cdot v_2}{\|v_2\|^2} \quad \frac{u \cdot v_k}{\|v_k\|^2}$$

$$P_W = C(C^T C)^{-1} C^T$$

$$C^T = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{bmatrix} \quad C = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$$

$$P_W = C D^{-1} C^T$$

Projected:

$$w = C D^{-1} C^T u$$

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Orthogonal Basis

Let $\{u_1, u_2, \cdots, u_k\}$ be a basis of a subspace W. How to transform $\{u_1, u_2, \cdots, u_k\}$ into an orthogonal basis $\{v_1, v_2, \cdots, v_k\}$?

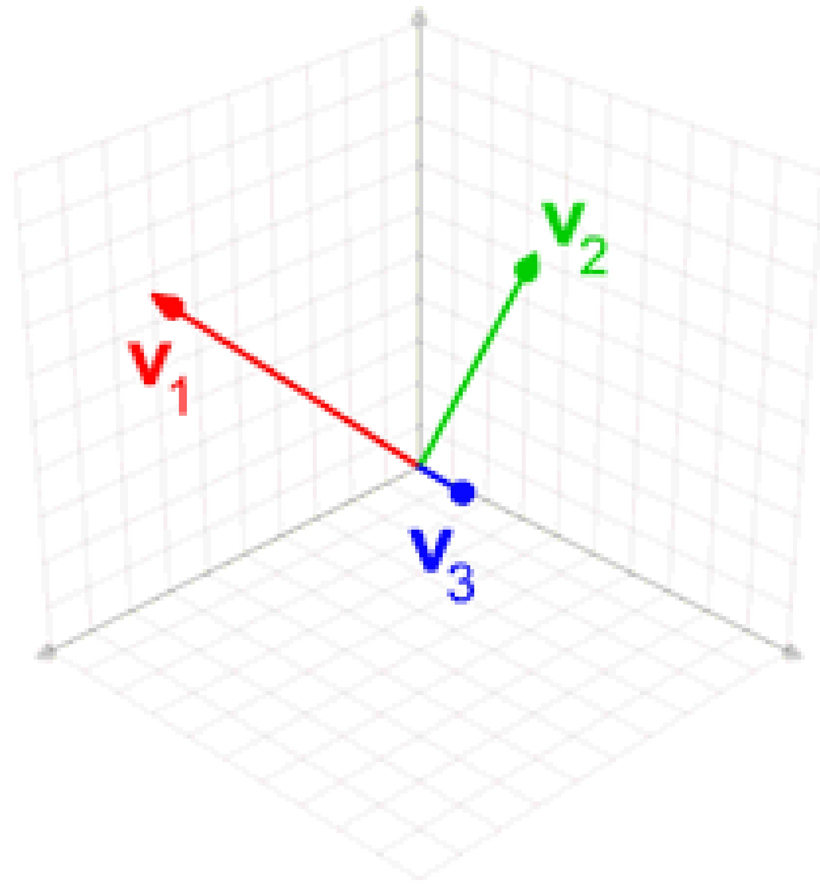
Gram-Schmidt Process

Then $\{v_1, v_2, \cdots, v_k\}$ is an orthogonal basis for W

$$\text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\} = \text{Span}\ \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_i\}$$

orthogonal

Visualization



https://www.youtube.com/watch?v=Ys28-Yq21B8

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2, \\
&\vdots \\
\mathbf{v}_k &= \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{||\mathbf{v}_{k-1}||^2}\mathbf{v}_{k-1}
\end{aligned}
$$

$\{u_1, u_2, \cdots, u_k\}$ be a basis of a subspace V

$$
\text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_i\} = \text{Span}\ \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_i\}
$$

none zero

orthogonal

The theorem holds for *i* = 1.

obviously

Assume the theorem holds for *i*=*n*, and consider the case for *n*+1.

$v_{n+1} \cdot v_j = 0$ (j < n + 1)

Example

$\mathcal{S}$ = {$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$} Is a basis for subspace W

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

(L.I. vectors)

Then $\mathcal{S}'$ = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is an orthogonal basis for W .

$\mathcal{S}''$ = {$\mathbf{v}_1$, $\mathbf{v}_2$, $4\mathbf{v}_3$} is also an orthogonal basis.

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1$$

,

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2$$